Resit Quantum Physics 1 – 2023/2024

Friday, February 2, 2023, 18:15 – 20:15

Read these instructions carefully. If you do not follow them your exam might be (partially) voided.

- This exam consists of 3 questions in 2 pages and a formula sheet at the end.
- The points for each question are indicated on the left side of the page.
- You have 2 hours to complete this exam.
- **Write your name and student number on *all* answer sheets that you turn in**.
- Start answering each exercise on a new page. It is ok to use front and back.
- Clearly write the total number of answer sheets that you turn in on the first page.
- Telephones, smart devices, and other electronic devices are **NOT** allowed.
- **This is a closed book exam**. Consulting reading material is **not** allowed.

32 pts **Question 1**

For this question, consider an electron subject to the infinite square well potential:

$V(x) = \begin{cases} 0, & \text{for } \; 0 < x < a \\ \infty, & \text{otherwise} \end{cases}$, where a is a positive constant.

10 pts **a)** Solve the Schrodinger equation to show that the eigenstates and eigenenergies of this system are, respectively, $\psi_n = \sqrt{\frac{2}{a}}\sin({}^{n\pi x}/_{a})$ and $E_n = {}^{n^2\pi^2\hbar^2}/_{2ma^2}$, with $n = 1, 2, 3, \ldots$. Explicitly mention the boundary conditions you use to solve the differential equation and be clear in all the steps you take towards the solution. Do not worry about the normalization.

12 pts **b)** Suppose that we prepare the electron in the state: $\phi(x) = \begin{cases} A, & \text{for } 0 < x < a \\ 0, & \text{otherwise} \end{cases}$.
Then, the width of the well is suddenly doubled, without letting the electron exchange energy with its surroundings. Find the time-dependent wavefunction $\Psi(x,t)$ in this system. *Tip*: Don't forget to normalize $\phi(x)$! *Tip2*: You can express $\Psi(x,t)$ as an infinite sum, but don't forget the coefficients.

10 pts **c)** For the wavefunction of (b), what is the probability of making an energy measurement and finding an energy $E = \frac{9\pi^2\hbar^2}{8ma^2}$?

33 pts **Question 2**

In this question we will discuss the (quantization of the) angular momentum $\vec{L}$.

10 pts **a)** Use mathematical arguments to demonstrate that it is possible to obtain functions which are eigenfunctions of L_z and L^2 simultaneously. In other words, show that L_z and L^2 are compatible observables. Remember that $L^2 = L_x^2 + L_y^2 + L_z^2$.

7 pts **b)** A system is in a quantum state such that its angular momentum quantum number $l = 1$. What are the possible values for measurements of L_z? And for L_x?

5 pts **c)** Take an eigenfunction of L_z given by $Y(\theta,\phi) = A\sin(\theta)\, e^{i\phi}$, where A is a constant. What is the value of m_l for this function? *Tip*: Look at the formula sheet for L_z.

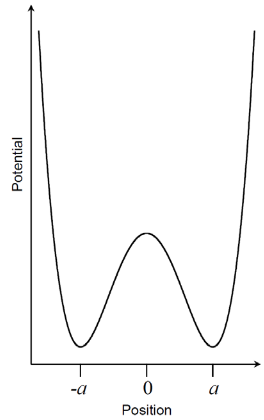
We now have two particles which are left to interact and end up in the quantum state: $|1,1\rangle|1,-1\rangle - |1,-1\rangle|1,1\rangle$, where we are using the notation $|l, m_l\rangle$ and the first ket refers to the state of the first particle and the second ket to the state of the second particle.

8 pts **d)** Is this state normalized? If not, what is the normalization constant?

3 pts **e)** Is this an entangled state? Why/why not? Briefly explain/calculate your answer.

35 pts **Question 3**

A good approximation for the potential landscape of a diatomic molecule (e.g. H_2) is shown below.



Here we will first assume that the barrier between the two atoms is high enough to prevent tunneling between the left and right wells. You can think of this situation as two isolated atoms, or highly localized electrons. We will only concern ourselves with the ground state of the separate wells and assume that all other energy states are so high in energy that they do not need to be considered.

We will denote the ground state of the left and right wells as $|\phi_L\rangle = \binom{1}{0}$ and $|\phi_R\rangle = \binom{0}{1}$, respectively. The energy of these states is given by: $\langle\phi_L|\hat{H}_0|\phi_L\rangle = \langle\phi_R|\hat{H}_0|\phi_R\rangle = \varepsilon$, and the terms $\langle\phi_L|\hat{H}_0|\phi_R\rangle = \langle\phi_R|\hat{H}_0|\phi_L\rangle = 0$.

2 pts **a)** Write the matrix which represents the Hamiltonian $\hat{H}_0$ written in the basis of $|\phi_L\rangle$ and $|\phi_R\rangle$.

7 pts **b)** Now assume that the barrier between the left and right sides of the well is reduced, allowing for cross-terms to exist: $\langle\phi_L|\hat{H}_1|\phi_R\rangle = \langle\phi_R|\hat{H}_1|\phi_L\rangle = T$, where T is a real and negative number. Assume that the eigenenergies of each separate well are not affected – i.e. the terms $\langle\phi_L|\hat{H}_1|\phi_L\rangle$ and $\langle\phi_R|\hat{H}_1|\phi_R\rangle$ are still equal to ε.
What is the matrix representation for this new Hamiltonian $\hat{H}_1$ (written on the $|\phi_L\rangle$ and $|\phi_R\rangle$ basis)? Calculate the ground and first excited states of $\hat{H}_1$, and their respective energies as a function of ε, T, $|\phi_L\rangle$ and $|\phi_R\rangle$.

Using a special experimental apparatus, we can make a measurement of the electron position and know if it is on the left or right side of the well. This measurement is described by the operator $\hat{A} = \begin{pmatrix} -a & 0 \\ 0 & a \end{pmatrix}$.

3 pts **c)** Are the eigenstates of $\hat{H}_0$ ($|\phi_L\rangle$ and $|\phi_R\rangle$) also eigenstates of $\hat{A}$? If so, what are the possible eigenvalues?

3 pts **d)** If we prepare a system subjected to the Hamiltonian with the lower barrier ($\hat{H}_1$) in its ground state, what are the possible results of making a measurement of $\hat{A}$ and with which probabilities?

5 pts **e)** We have performed a measurement of $\hat{A}$ and found $-a$. What is the state of the system just after this measurement? Express this state in terms of the ground and excited states of $\hat{H}_1$: $|\phi_g\rangle$ and $|\phi_e\rangle$.

15 pts **f)** Calculate how $\langle\hat{A}(t)\rangle$ evolves in time after the measurement. Explain in your own words what this calculation represents. You will need the time-evolution operator $\hat{U} = e^{-i\hat{H}t/\hbar}$.

Useful formulas:

Schrodinger equation $i\hbar\frac{\partial\Psi}{\partial t} = H\Psi$

Time-independent Schrodinger equation $H\psi = E\psi \qquad \Psi = \psi e^{-iEt/\hbar}$

Hamiltonian operator $H = -\frac{\hbar^2}{2m}\nabla^2 + V$

Momentum operator $p = -i\hbar\,\nabla$

De Broglie wavelength $\lambda = h/p$

Time-dependence of expectation value $\frac{d\langle Q\rangle}{dt} = \frac{i}{\hbar}\langle[H,Q]\rangle + \langle\frac{\partial Q}{\partial t}\rangle$

Generalized uncertainty principle $\sigma_A\sigma_B \geq \left|\frac{1}{2i}\langle[A,B]\rangle\right|$

Heisenberg Uncertainty principle $\sigma_x\sigma_p \geq \hbar/2$

Canonical commutator $[x,p] = i\hbar$

Angular momentum $[L_x, L_y] = i\hbar L_z \;;\; [L_y, L_z] = i\hbar L_x \;;\; [L_z, L_x] = i\hbar L_y$

In spherical coordinates:

$L_z = -i\hbar\; {}^{\partial}/_{\partial\phi}$

$L_x = -i\hbar\left(-\sin\phi\; {}^{\partial}/_{\partial\theta} - \cos\phi\cot\theta\; {}^{\partial}/_{\partial\phi}\right)$

$L_y = -i\hbar\left(+\cos\phi\; {}^{\partial}/_{\partial\theta} - \sin\phi\cot\theta\; {}^{\partial}/_{\partial\phi}\right)$

Pauli matrices $\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \;;\; \sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \;;\; \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

Trigonometric relations

$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$

$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$

$\sin(90^\circ \pm \theta) = \cos(\theta)$

$\cos(90^\circ \pm \theta) = \mp\sin(\theta)$